Chapter 3

Vanishing and Kodaira embedding theorem

3.1 Bochner methods and vanishing theorem

For a vector bundle E over a Riemannian manifold M with a connection ∇^E, by taking a locally orthonormal basis, the usual Bochner Laplacian Δ^E is defined by

$$\Delta^E = -\sum_{j=1}^{\dim_{\mathbb{R}} M} \left((\nabla^E_{e_j})^2 - \nabla^E_{\nabla^{TX}_{e_j} e_j} \right). \tag{3.1.1}$$

We assume that the vector bundle E admits a Euclidean metric if it is real or a Hermitian metric if it is complex. We denote the corresponding metric by $\langle \cdot, \cdot \rangle$. We assume that the connection ∇^E preserves the metric on E.

For $s_1, s_2 \in \mathscr{C}^\infty(M, E)$ with compact support, we have

$$\begin{aligned} \int_M \langle \Delta^E s_1, s_2 \rangle dv &= \sum_{j=1}^{\dim_{\mathbb{R}} M} \int_M \langle \nabla^E_{e_j} s_1, \nabla^E_{e_j} s_2 \rangle dv - \int_M \operatorname{tr}(\nabla \alpha) dv \\ &= \sum_{j=1}^{\dim_{\mathbb{R}} M} \int_M \langle \nabla^E_{e_j} s_1, \nabla^E_{e_j} s_2 \rangle dv = \int_M \langle s_1, \Delta^E s_2 \rangle dv, \end{aligned} \tag{3.1.2}$$

where $\alpha(Y) = \langle \nabla^E_Y s_1, s_2 \rangle$.

Lemma 3.1.1. *Let V be a real vector space with basis e_i. For any $A \in \operatorname{End}(V)$, there exists a unique endomorphism $\lambda(A)$, which is called the **derivation**, on ΛV, such that it coincides with A on V and satisfies the Leibniz's* *rules:*

$$\lambda(A)(a \wedge b) = A(a) \wedge b + a \wedge A(b), \tag{3.1.3}$$

where $a, b \in \Lambda V$. *Explicitly, it is given by*

$$\lambda(A) = \langle e^j, Ae_k \rangle e_j \wedge i_{e_k}. \tag{3.1.4}$$

Proof. The uniqueness is obvious. We only need to prove that (3.1.4) is a derivation. Firstly, for $e_k \in V$, we have $\lambda(A)e_k = \langle e^j, Ae_k\rangle e_j = Ae_k$. Secondly, the operator $e_j \wedge i_{e_k}$ satisfies the Leibniz's rule (3.1.3).

The proof of our lemma is completed. □

Theorem 3.1.2 (Weitzenböck's formula)**.** *Let R be the curvature of the Levi-Civita connection on TM. Then*

$$(d + d^*)^2 = \Delta^{\Lambda T^*M} - \sum_{ijkl} R_{ijkl} e^k \wedge i_{e_l} e^i \wedge i_{e_j}. \tag{3.1.5}$$

In particular, on the space of one forms, we have

$$\Delta_{\mathbb{R}} = (d + d^*)^2 = \Delta^{\Lambda T^*M} + \mathrm{Ric}(e_i, e_j) e^i \wedge i_{e_j}. \tag{3.1.6}$$

Proof. Let $\nabla^{\Lambda T^*M}$ be the connection on ΛT^*M induced by the Levi-Civita connection ∇. Let $R^{\Lambda T^*M}$ be the curvature of $\nabla^{\Lambda T^*M}$. From (2.2.61) and (2.2.62), we have

$$d = e^j \wedge \nabla^{\Lambda T^*M}_{e_j}, \quad d^* = -i_{e_j} \nabla^{\Lambda T^*M}_{e_j}. \tag{3.1.7}$$

Since the formulas (3.1.5) and (3.1.6) do not depend on the choice of the locally orthonormal coordinates. We choose the normal coordinates. Notice that

$$e^i \wedge i_{e_j} + i_{e_j} e^i \wedge = \delta_{ij} \,\mathrm{Id}\,. \tag{3.1.8}$$

We have

$$\begin{aligned} dd^* + d^*d &= -e^i \wedge i_{e_j} \nabla^{\Lambda T^*M}_{e_i} \nabla^{\Lambda T^*M}_{e_j} - i_{e_j} e^i \wedge \nabla^{\Lambda T^*M}_{e_j} \nabla^{\Lambda T^*M}_{e_i} \\ &= -\nabla^{\Lambda T^*M}_{e_i} \nabla^{\Lambda T^*M}_{e_i} - R^{\Lambda T^*M}(e_i, e_j) e^i \wedge i_{e_j}. \end{aligned} \tag{3.1.9}$$

Let R^{TM} be the curvature of the Levi-Civita connection ∇. It is easy to see that $R^{\Lambda T^*M}$ is the derivation of R^{T^*M}. By (3.1.4), we have

$$R^{\Lambda T^*M} = \langle e_k, R^{T^*M} e^l \rangle e^k \wedge i_{e_l} = \langle R^{TM} e_l, e_k \rangle e^k \wedge i_{e_l}. \tag{3.1.10}$$

Combining (3.1.9) and (3.1.10), we have

$$dd^* + d^*d = \Delta^{\Lambda T^*M} - R_{ijkl} e^k \wedge i_{e_l} e^i \wedge i_{e_j}. \tag{3.1.11}$$

From (3.1.8), we have

$$\begin{aligned} R_{ijkl} e^k \wedge i_{e_l} e^i \wedge i_{e_j} &= -R_{ijkl} e^k \wedge e^i \wedge i_{e_l} i_{e_j} + R_{ijki} e^k \wedge i_{e_j} \\ &= -R_{ijkl} e^k \wedge e^i \wedge i_{e_l} i_{e_j} - \mathrm{Ric}(e_i, e_j) e^i \wedge i_{e_j}. \end{aligned} \tag{3.1.12}$$

Notice that the first term on the right-hand side vanishes on one forms. Then we get (3.1.6).

The proof of our theorem is completed. □

Definition 3.1.3. A function (resp. a twofold symmetric covariant tensor, etc) on a manifold is **quasi-positive** if it is everywhere nonnegative (resp. positive semi-definite, etc) and is positive (resp. positive definite, etc) at a point. **Quasi-negativity** is dually defined.

Theorem 3.1.4 (Bochner 1946)**.** *For a compact orientable Riemannian manifold M of nonnegative Ricci curvature, its first Betti number $b_1 \leq \dim M$, with the upper bound attained by the flat torus. If the Ricci curvature is quasi-positive, then $b_1 = 0$.*

Proof. From (3.1.2), for any $\alpha \in \Omega^1(M)$, then

$$\int_M \langle \Delta^{\Lambda T^*M} \alpha, \alpha \rangle dv = \sum_{j=1}^{\dim_{\mathbb{R}} M} \|\nabla_{e_j}^{\Lambda T^*M} \alpha\|_{L_2}^2 \geq 0. \tag{3.1.13}$$

If the Ricci curvature is quasi-positive, there exists $x \in M$ such that $\alpha = 0$ on a neighbourhood of x. Since

$$\int_M \langle \mathrm{Ric}(e_i, e_j) e^i \wedge i_{e_j} \alpha, \alpha \rangle dv \geq 0, \tag{3.1.14}$$

by (3.1.6) and (3.1.13), we have $\nabla^{\Lambda T^*M} \alpha = 0$. So $\alpha \equiv 0$. Thus $\ker \Delta_{\mathbb{R}} = 0$. From the Hodge theorem 2.2.6, we have $b_1 = 0$.

If the Ricci curvature is nonnegative, we have

$$\int_M \langle \mathrm{Ric}(e_i, e_j) e^i \wedge i_{e_j} \alpha, \alpha \rangle dv \geq 0. \tag{3.1.15}$$

If $\alpha \in \ker \Delta_{\mathbb{R}}$, from (3.1.6), (3.1.13) and (3.1.15), we have $\nabla^{\Lambda T^*M} \alpha = 0$. For any $x \in M$, we have

$$b_1 \leq \dim_{\mathbb{R}} \{\alpha_x : \nabla^{\Lambda T^*M} \alpha = 0\} = \dim_{\mathbb{R}} M. \tag{3.1.16}$$

Notice that for torus T^n, $H^n(T^n, \mathbb{R}) = H^1(S^1, \mathbb{R})^{\otimes n} = \mathbb{R}^n$. Thus the proof of our theorem is completed. □

Now we consider the Kähler case.

Let (M,ω) be a compact orientable Kähler manifold. Let E be a Hermitian holomorphic vector bundle over M with Hermitian connection ∇^E. We simply denote by $\Delta^{0,\cdot}$ the Laplacian with respect to the connection $\nabla^{\Lambda T^{0,1}M\otimes E}$ induced by the connections $\nabla^{T^{(0,1)}M}$ and ∇^E. Recall that $K_M^* = \Lambda^n(T^{1,0}M)$ and

$$\operatorname{tr}\left[R^{T^{1,0}M}\right] = R^{K_M^*} = -\sqrt{-1}\operatorname{Ric}_\omega . \tag{3.1.17}$$

Theorem 3.1.5 (Bochner-Kodaira). *Let E be a Hermitian holomorphic vector bundle over the Kähler manifold M. In a local holomorphic coordinate system,*

$$\begin{aligned}\square^E = (\bar\partial^E + \bar\partial^{E,*})^2 = \frac{1}{2}\Delta^{0,\cdot} - \frac{1}{2}R^E(\theta_i,\bar\theta_i)\\ + \left(R^E + \frac{1}{2}\operatorname{tr}\left[R^{T^{1,0}M}\right]\right)(\theta_j,\bar\theta_k)\bar\theta^k\wedge i_{\bar\theta_j}.\end{aligned} \tag{3.1.18}$$

Proof. By Theorem 1.2.15, we could choose the normal holomorphic coordinates. In this coordinates around $x\in M$, we have $[\nabla, i_{\bar\theta_k}] = [\nabla, \bar\theta^k\wedge] = 0$ and $[\bar\theta_j,\theta_k] = \nabla_{\bar\theta_j}\theta_k - \nabla_{\theta_k}\bar\theta_j = 0$ at x.

By (2.2.72) and (2.2.73), $\bar\partial^E = \bar\theta^j\wedge\nabla^{\Lambda T^{(0,1)}M\otimes E}_{\bar\theta_j}$ and $\bar\partial^{E,*} = -i_{\bar\theta_j}\nabla^{\Lambda T^{(0,1)}M\otimes E}_{\theta_j}$. We simply denote by $\nabla^{0,\cdot} := \nabla^{\Lambda T^{(0,1)}M\otimes E}$ Thus

$$\begin{aligned}\bar\partial^E\bar\partial^{E,*} + \bar\partial^{E,*}\bar\partial^E &= -\bar\theta^j\wedge i_{\bar\theta_k}\nabla^{0,\cdot}_{\bar\theta_j}\nabla^{0,\cdot}_{\theta_k} - i_{\bar\theta_k}\bar\theta^j\wedge\nabla^{0,\cdot}_{\theta_k}\nabla^{0,\cdot}_{\bar\theta_j}\\ &= -(\bar\theta^j\wedge i_{\bar\theta_k} + i_{\bar\theta_k}\bar\theta^j\wedge)\nabla^{0,\cdot}_{\theta_k}\nabla^{0,\cdot}_{\bar\theta_j} - \bar\theta^j\wedge i_{\bar\theta_k}\left(\nabla^{0,\cdot}_{\bar\theta_j}\nabla^{0,\cdot}_{\theta_k} - \nabla^{0,\cdot}_{\theta_k}\nabla^{0,\cdot}_{\bar\theta_j}\right)\\ &= -\nabla^{0,\cdot}_{\theta_j}\nabla^{0,\cdot}_{\bar\theta_j} + R^E(\theta_k,\bar\theta_j)\bar\theta^j i_{\bar\theta_k} + R^{T^{0,\cdot *}M}(\theta_k,\bar\theta_j)\bar\theta^j i_{\bar\theta_k}.\end{aligned} \tag{3.1.19}$$

By (2.2.71),

$$\sum_{i=1}^{2n}\nabla^{0,\cdot}_{e_i}\nabla^{0,\cdot}_{e_i} = \sum_{i=1}^{n}\left(\nabla^{0,\cdot}_{\theta_i}\nabla^{0,\cdot}_{\bar\theta_i} + \nabla^{0,\cdot}_{\bar\theta_i}\nabla^{0,\cdot}_{\theta_i}\right) = 2\sum_{i=1}^{n}\nabla^{0,\cdot}_{\theta_i}\nabla^{0,\cdot}_{\bar\theta_i} - \sum_{i=1}^{n}R^{0,\cdot}(\theta_i,\bar\theta_i). \tag{3.1.20}$$

Since we choose the normal coordinates for Kähler manifold, by (2.2.71), $\sum_{i=1}^{2n}\nabla^{TX}_{e_i}e_i = \sum_{i=1}^{n}\nabla^{TX}_{\theta_i}\bar\theta_i + \sum_{i=1}^{n}\nabla^{TX}_{\bar\theta_i}\theta_i = 0$. So

$$-\nabla^{0,\cdot}_{\theta_j}\nabla^{0,\cdot}_{\bar\theta_j} = \frac{1}{2}\Delta^{0,\cdot} - \frac{1}{2}R^E(\theta_i,\bar\theta_i) - \frac{1}{2}R^{T^{0,*}M}(\theta_i,\bar\theta_i). \tag{3.1.21}$$

From Lemma 3.1.1,

$$R^{\Lambda T^{0,1*}M} = \langle \bar{\theta}_l, R^{T^{0,1*}M}\bar{\theta}^s\rangle \bar{\theta}^l \wedge i_{\bar{\theta}_s} = g(R\theta_s, \bar{\theta}_l)\bar{\theta}^l \wedge i_{\bar{\theta}_s}. \tag{3.1.22}$$

Thus

$$\begin{aligned} R^{T^{0,\cdot*}M}(\theta_k, \bar{\theta}_j)\bar{\theta}^j i_{\bar{\theta}_k} - \frac{1}{2}R^{T^{0,*}M}(\theta_i, \bar{\theta}_i) \\ = -R_{k\bar{j}s\bar{l}}\bar{\theta}^l \wedge i_{\bar{\theta}_s}\bar{\theta}^j \wedge i_{\bar{\theta}_k} + \frac{1}{2}R_{j\bar{j}s\bar{l}}\bar{\theta}^l \wedge i_{\bar{\theta}_s} \end{aligned} \tag{3.1.23}$$

By Bianchi Identity, $R_{k\bar{j}s\bar{l}} + R_{sk\bar{j}\bar{l}} + R_{\bar{j}sk\bar{l}} = 0$. Since $R_{sk\bar{j}\bar{l}} = 0$, we have

$$R_{k\bar{j}s\bar{l}} = R_{s\bar{j}k\bar{l}}. \tag{3.1.24}$$

As in (3.1.8), we have

$$\bar{\theta}^i \wedge i_{\bar{\theta}_j} + i_{\bar{\theta}_j}\bar{\theta}^i \wedge = \delta_{ij}\,\mathrm{Id}\,. \tag{3.1.25}$$

So

$$\begin{aligned} R_{k\bar{j}s\bar{l}}\bar{\theta}^l \wedge i_{\bar{\theta}_s}\bar{\theta}^j \wedge i_{\bar{\theta}_k} &= R_{k\bar{j}s\bar{l}}\bar{\theta}^l \wedge i_{\bar{\theta}_k}\bar{\theta}^j \wedge i_{\bar{\theta}_s} \\ &= -R_{k\bar{j}s\bar{l}}\bar{\theta}^l \wedge i_{\bar{\theta}_s}\bar{\theta}^j \wedge i_{\bar{\theta}_k} + R_{k\bar{j}j\bar{l}}\bar{\theta}^l \wedge i_{\bar{\theta}_k} + R_{j\bar{j}s\bar{l}}\bar{\theta}^l \wedge i_{\bar{\theta}_s} \end{aligned} \tag{3.1.26}$$

It implies

$$-R_{k\bar{j}s\bar{l}}\bar{\theta}^l \wedge i_{\bar{\theta}_s}\bar{\theta}^j \wedge i_{\bar{\theta}_k} = -R_{j\bar{j}s\bar{l}}\bar{\theta}^l \wedge i_{\bar{\theta}_s}. \tag{3.1.27}$$

Recall that in (2.1.56), we get

$$\mathrm{tr}[R^{T^{1,0}M}] = R^{K_M^*} = \mathrm{Ric}_\omega\,. \tag{3.1.28}$$

Since

$$-R_{j\bar{j}s\bar{l}} = -R_{s\bar{l}j\bar{j}} = g(R(\theta_s, \bar{\theta}_l)\theta_j, \bar{\theta}_j) = \mathrm{tr}[R^{T^{1,0}M}](\theta_s, \bar{\theta}_l), \tag{3.1.29}$$

We obtain the theorem.

Our proof of the theorem is completed. □

Theorem 3.1.6. *On a compact Kähler manifold M with quasi-positive bisectional curvature, we have $h^{1,1} = 1$.*

Proof. In this case, $E = \Lambda(T^{*(1,0)}M)$. We have

$$R^{\Lambda(T^{*(1,0)}M)}(\theta_i, \bar{\theta}_i) = R_{i\bar{i}l\bar{s}}\theta^l \wedge i_{\theta_s} \tag{3.1.30}$$

and

$$R^{\Lambda(T^{*(1,0)}M)}(\theta_j, \bar{\theta}_k)\bar{\theta}^k \wedge i_{\bar{\theta}_j} = R_{j\bar{k}l\bar{s}}\theta^l \wedge i_{\theta_s}\bar{\theta}^k \wedge i_{\bar{\theta}_j}. \tag{3.1.31}$$

Thus by (3.1.18),

$$\Box^E - \frac{1}{2}\Delta^{0,\cdot} = -\frac{1}{2}R_{i\bar{i}l\bar{s}}\theta^l \wedge i_{\theta_s} + R_{j\bar{k}l\bar{s}}\theta^l \wedge i_{\theta_s}\bar{\theta}^k \wedge i_{\bar{\theta}_j} - \frac{1}{2}R_{j\bar{k}i\bar{i}}\bar{\theta}^k \wedge i_{\bar{\theta}_j}. \tag{3.1.32}$$

For harmonic real $(1,1)$-form α, if we write $\alpha = \sum_{i,j}\alpha_{ij}\theta^i \wedge \bar{\theta}^j$, we have

$$\sum_{i,j}\alpha_{ij}\theta^i \wedge \bar{\theta}^j = \alpha = \bar{\alpha} = \overline{\alpha_{ij}}\bar{\theta}^i \wedge \theta^j = -\sum_{i,j}\overline{\alpha_{ji}}\theta^i \wedge \bar{\theta}^j. \tag{3.1.33}$$

Thus after an orthogonal transform, we could assume that α could be written as $\alpha = \sum_i \sqrt{-1}\alpha_i\theta^i \wedge \bar{\theta}^i$ where α_i is a real-valued function. From (3.1.32), we have

$$\frac{1}{2}\Delta^{0,\cdot}\alpha = \frac{\sqrt{-1}}{2}R_{i\bar{i}l\bar{k}}\alpha_k\theta^l \wedge \bar{\theta}^k - \sqrt{-1}R_{i\bar{k}l\bar{i}}\alpha_i\theta^l \wedge \bar{\theta}^k + \frac{\sqrt{-1}}{2}R_{l\bar{k}i\bar{i}}\alpha_l\theta^l \wedge \bar{\theta}^k. \tag{3.1.34}$$

Taking the conjuation,

$$\begin{aligned}\sqrt{-1}R_{l\bar{k}i\bar{i}}\alpha_l\theta^l \wedge \bar{\theta}^k &= \sqrt{-1}R_{k\bar{l}i\bar{i}}\alpha_k\theta^k \wedge \bar{\theta}^l = -\sqrt{-1}R_{k\bar{l}i\bar{i}}\alpha_k\bar{\theta}^l \wedge \theta^k \\ &= \sqrt{-1}R_{\bar{k}l\bar{i}i}\alpha_k\theta^l \wedge \bar{\theta}^k = \sqrt{-1}R_{l\bar{k}i\bar{i}}\alpha_k\theta^l \wedge \bar{\theta}^k.\end{aligned} \tag{3.1.35}$$

So we have

$$\frac{1}{2}\Delta^{0,\cdot}\alpha = \sqrt{-1}R_{i\bar{i}l\bar{k}}\alpha_k\theta^l \wedge \bar{\theta}^k - \sqrt{-1}R_{i\bar{k}l\bar{i}}\alpha_i\theta^l \wedge \bar{\theta}^k. \tag{3.1.36}$$

From (3.1.2) and (3.1.24), for harmonic real $(1,1)$-form α, we have

$$\begin{aligned}\sum_i \|\nabla^{0,\cdot}_{e_i}\alpha\|^2_{L_2} &= -\int_M (2R_{i\bar{i}k\bar{k}}\alpha_k^2 + 2R_{i\bar{k}k\bar{i}}\alpha_i\alpha_k)dv \\ &= -\int_M R_{i\bar{i}k\bar{k}}(\alpha_i - \alpha_k)^2 dv.\end{aligned} \tag{3.1.37}$$

If the bisectional curvature is quasi-positive, we have $\alpha_i = \alpha_k$ for any i, k. Thus $\alpha = \phi \cdot \omega$, where ϕ is a real-valued function. Since $\nabla^{0,\cdot}_{e_i}\alpha = 0$, we see that ϕ is a constant. Thus $h^{1,1} = 1$.

The proof of our theorem is completed. □

In general, a markable extension of Theorem 1.3.16 (Siu-Yau, Mori) exists.

Theorem 3.1.7 (Mok 1988)**.** *A compact Kähler manifold with quasi-positive bisectional curvature is biholomorphic to complex projective space.*

Theorem 3.1.8. *For negative holomorphic line bundle L over complex manifold M, we have $H^0(M, L) = 0$ for $p > 0$.*

Proof. Take $E = L$ in (3.1.18). If L is negative, by Definition 2.1.18, we have $R^L(\theta_i, \bar{\theta}_i) = \sqrt{-1} R^L(\theta_i, J\bar{\theta}_i) < 0$. Following the same arguments, we get our theorem. □

Theorem 3.1.9. *Let (M, ω) be a compact Kähler manifold such that* Ric_ω *is quasi-positive. Then $h^{p,0} = 0$ for any $p > 0$.*

Proof. Let α be a harmonic $(p, 0)$-form. Then by Theorem 3.1.5and (3.1.22),

$$\Delta^{0,\cdot}\alpha = R^{\Lambda(T^{*(1,0)}M)}(\theta_i, \bar{\theta}_i)\alpha = R_{i\bar{i}l\bar{s}}\theta^l \wedge i_{\theta_s}\alpha \tag{3.1.38}$$

From Definition 2.1.18, if Ric_ω is quasi-positive, then $\mathrm{Ric}_\omega(\cdot, J\cdot)$ is quasi-positive. From (1.3.17),

$$\mathrm{Ric}_\omega(\theta_l, J\bar{\theta}_s) = -\sqrt{-1}\,\mathrm{Ric}_\omega(\theta_l, \bar{\theta}_s) = R_{l\bar{s}i\bar{i}} = -R_{i\bar{i}l\bar{s}}. \tag{3.1.39}$$

So for any l, s, $R_{i\bar{i}l\bar{s}}$ is quasi-negative. So $\int_M \langle R_{i\bar{i}l\bar{s}}\theta^l \wedge i_{\theta_s}\alpha, \alpha\rangle < 0$. Since $\int_M \langle \Delta^{0,\cdot}\alpha, \alpha\rangle \geq 0$, we see that $\alpha = 0$.

The proof of our theorem is completed. □

Corollary 3.1.10 (Kobayashi)**.** *A compact connected Kähler manifold with positive Ricci curvature is simply connected.*

Proof. Since $h^{p,q} = h^{q,p}$, we see that for any $p > 0$, $h^{0,p} = 0$. Notice that the only holomorphic functions on connected compact complex manifold are constants. Thus $h^{0,0} = 1$. So $\chi_0(M) = \sum_{p=0}^n (-1)^p h^{0,p} = 1$.

From the Myer's theorem, since M is compact and the Ricci tensor has the positive lower bound, the fundamental group $\pi_1(M)$ is finite. Let $\tilde{M}$ be the universal cover of M. Then $\tilde{M}$ is compact with positive Ricci curvature. It implies that $\chi_0(\tilde{M}) = 1$. We lift the geometric structure of M onto $\tilde{M}$. Then we have

$$\int_{\tilde{M}} \mathrm{Td}(T^{(1,0)}\tilde{M}) = |\pi_1(M)| \int_M \mathrm{Td}(T^{(1,0)}M). \tag{3.1.40}$$

From the Hirzebruch-Riemann-Roch theorem,

$$\int_{\tilde{M}} \mathrm{Td}(T^{(1,0)}\tilde{M}) = \chi_0(\tilde{M}) = 1 = \chi_0(M) = \int_M \mathrm{Td}(T^{(1,0)}M). \tag{3.1.41}$$

So we get $\pi_1(M) = 1$.

The proof of our corollary is completed. □

Corollary 3.1.11. *Fano manifolds are simply connected.*

Proof. Let M be a Fano manifold. Then $c_1(M) > 0$. From the Calabi-Yau theorem 2.1.17, there exists a Kähler form ω such that $\mathrm{Ric}_\omega > 0$.

The proof is completed. □

Theorem 3.1.12 (Nakano's inequality)**.** *For holomorphic vector bundle E over a compact Kähler manifold M, and any $s \in \Omega^{\cdot,\cdot}(M, E)$,*

$$\langle \Box^E s, s\rangle_E \geq \langle [\sqrt{-1}R^E, \Lambda]s, s\rangle_E. \tag{3.1.42}$$

Proof. By Bochner-Kodaira-Nakano formula Theorem 2.2.23,

$$\begin{aligned}\langle \Box^E s, s\rangle_E &= \|\bar\partial^E s\|^2_{L^2} + \|\bar\partial^{E,*} s\|^2_{L^2} \\ &= \|(\nabla^E)^{1,0} s\|^2_{L^2} + \|(\nabla^E)^{1,0*} s\|^2_{L^2} + \langle \sqrt{-1}[R^E, \Lambda]s, s\rangle_E.\end{aligned} \tag{3.1.43}$$

The proof of our theorem is completed. □

Theorem 3.1.13. *Let M be a compact complex manifold of complex dimension n and L be a positive holomorphic line bundle over M. Then*

(a) (Kodaira vanishing theorem) if $q > 0$

$$H^q(M, L \otimes K_M) = 0; \tag{3.1.44}$$

(b) (Nakano vanishing theorem) if $p + q > n$,

$$H^{p,q}(M, L) = 0. \tag{3.1.45}$$

Proof. Since L is positive, $\omega = \frac{\sqrt{-1}}{2\pi} R^L$ is a positive $(1,1)$-form. Let g^{TX} be the associated Kähler metric on TM. As $\omega = \sqrt{-1}\theta^i \wedge \bar\theta^i$, by (2.2.92), we have

$$[\omega, \Lambda] = \theta^i \wedge i_{\theta_i} - i_{\bar\theta_i} \bar\theta^i \wedge . \tag{3.1.46}$$

Thus for $s \in \Omega^{p,q}(M, L)$, we have

$$[\omega, \Lambda]s = (p + q - n)s. \tag{3.1.47}$$

Then the Nakano's inequality Theorem 3.1.12 implies that if $\Box^L s = 0$, it follows that $s = 0$ whenever $p + q > n$. By Hodge theorem for holomorphic vector bundle $\Lambda^p(T^{*(1,0)}M) \otimes L$, we get (b). (a) is a case of (b) for $p = n$.

The proof of our theorem is completed. □

Theorem 3.1.14 (Kodaira-Serre vanishing theorem). *Let L be a positive holomorphic line bundle and E be a holomorphic vector bundle. Then there exists $p_0 > 0$ such that for any $p \geq p_0$,*

$$H^q(M, L^p \otimes E) = 0 \quad \textit{for any } q > 0. \tag{3.1.48}$$

Proof. From (3.1.19), for any $s \in \oplus_{p\geq 1}\Omega^{0,q}(M, L^p \otimes E)$,

$$\begin{aligned}\langle \square^{L^p\otimes E} s, s\rangle &= \|\bar{\partial}^{L^p\otimes E} s\|^2_{L_2} + \|\bar{\partial}^{L^p\otimes E,*} s\|^2_{L_2} = \sum_{i=1}^{n} \|\nabla^{0,\cdot}_{\bar{\theta}_i} s\|^2_{L_2} \\ &\quad + \langle R^{L^p\otimes E}(\theta_j, \bar{\theta}_k)\bar{\theta}^k \wedge i_{\bar{\theta}_j} s, s\rangle + \langle R^{\Lambda T^{*(0,1)}M}(\theta_j, \bar{\theta}_k)\bar{\theta}^k \wedge i_{\bar{\theta}_j} s, s\rangle \\ &\geq p\langle R^L(\theta_j, \bar{\theta}_k)\bar{\theta}^k \wedge i_{\bar{\theta}_j} s, s\rangle + \langle R^{\Lambda T^{*(0,1)}M\otimes E}(\theta_j, \bar{\theta}_k)\bar{\theta}^k \wedge i_{\bar{\theta}_j} s, s\rangle. \end{aligned} \tag{3.1.49}$$

We identify the two form R^L with the Hermitian matrix $\dot{R}^L \in \mathrm{End}(T^{(1,0)}M)$ such that for $X, Y \in T^{(1,0)}M$,

$$R^L(X, \overline{Y}) = \langle \dot{R}^L X, \overline{Y}\rangle. \tag{3.1.50}$$

After an orthogonal transform, we could assume that

$$\dot{R}^L(x) = \mathrm{diag}(a_1(x), \cdots, a_n(x)) \in \mathrm{End}(T^{(1,0)}_x M). \tag{3.1.51}$$

Since L is positive, for any $x \in M$ and $1 \leq j \leq n$, $a_j(x) > 0$. So there exists $C_0 > 0$ such that

$$\langle R^L(\theta_j, \bar{\theta}_k)\bar{\theta}^k \wedge i_{\bar{\theta}_j} s, s\rangle = \langle \sum_j a_j(x)\bar{\theta}^j \wedge i_{\bar{\theta}_j} s, s\rangle \geq C_0\|s\|^2_{L_2} \tag{3.1.52}$$

Thus from (3.1.49) and (3.1.52), there exists $C_1 > 0$ such that

$$\langle \square^{L^p\otimes E} s, s\rangle \geq (C_0 p - C_1)\|s\|^2_{L_2}. \tag{3.1.53}$$

If p is taken large enough such that $C_0 p - C_1 > 0$, we have $\ker \square^{L^p\otimes E} = 0$. From the Hodge theory, we obtain the Kodaira-Serre vanishing theorem. □

For complex manifold, we also have the corresponding Bochner-Kodaira type formula. We only state here without proof.

Let M be a compact complex manifold and E be a holomorphic vector bundle over M. There are two natural connections: Levi-Civita connection ∇ and Chern connection $\widetilde{\nabla}$. If the manifold is Kähler, they are equal.

Set

$$S := \widetilde{\nabla} - \nabla. \tag{3.1.54}$$

Take $S^B \in \Omega^1(M, \mathrm{End}(TM))$ such that

$$g(S^B(U)V, W) = \frac{\sqrt{-1}}{2}\big((\partial - \bar{\partial})\omega\big)(U, V, W) \tag{3.1.55}$$

for any $U, V, W \in TM$. The **Bismut connection** ∇^B on TM is defined by

$$\nabla^B := \nabla + S^B = \widetilde{\nabla} + S^B - S. \tag{3.1.56}$$

Remark that the Bismut connection preserves the complex structure. Thus it induces a natural connection ∇^B on $\Lambda(T^{*(0,1)}M)$. Let $\nabla^{B,\Lambda^{0,\cdot}}$, $\nabla^{B,\Lambda^{0,\cdot}\otimes E}$ be the connections on $\Lambda(T^{*(0,1)}M)$, $\Lambda(T^{*(0,1)}M) \otimes E$ defined by

$$\nabla^{B,\Lambda^{0,\cdot}} = \nabla^B + \langle S(\cdot)\theta_j, \bar{\theta}_j\rangle, \tag{3.1.57}$$

$$\nabla^{B,\Lambda^{0,\cdot}\otimes E} = \nabla^{B,\Lambda^{0,\cdot}} \otimes 1 + 1 \otimes \nabla^E. \tag{3.1.58}$$

For any $v \in TM$ with the decomposition $v = v^{1,0} + v^{0,1} \in T^{(1,0)}M \oplus T^{(0,1)}M$, let $\bar{v}^{1,0,*}$ be the metric dual of $v^{1,0}$. Then we set

$$c(v) := \sqrt{2}(\bar{v}^{1,0,*} \wedge - i_{v^{0,1}}) \in \mathrm{End}(\Lambda(T^{*(0,1)}M)). \tag{3.1.59}$$

We verify easily that for $U, V \in TM$,

$$c(U)c(V) + c(V)c(U) = -2g(U, V). \tag{3.1.60}$$

For a skew-adjoint endomorphism A of TM, from (3.1.59), we could calculate that

$$\begin{aligned}\frac{1}{4}g(Ae_i, e_j)c(e_i)c(e_j) = &-\frac{1}{2}g(A\theta_j, \bar{\theta}_j) + g(A\theta_l, \bar{\theta}_m)\bar{\theta}^m \wedge i_{\bar{\theta}_l} \\ &+ \frac{1}{2}g(A\theta_l, \theta_m)i_{\bar{\theta}_l}i_{\bar{\theta}_m} + \frac{1}{2}g(A\bar{\theta}_l, \bar{\theta}_m)i_{\bar{\theta}_l}i_{\bar{\theta}_m}\bar{\theta}^l \wedge \bar{\theta}^m \wedge .\end{aligned} \tag{3.1.61}$$

For $i_1 < \cdots < i_j$, we define

$$^c(e^{i_1} \wedge \cdots \wedge e^{i_j}) = c(e_{i_1}) \cdots c(e_{i_j}). \tag{3.1.62}$$

Then by extending $\mathbb{C}$-linearly, ^cA is defined for any $A \in \Lambda(T^*M \otimes_{\mathbb{R}} \mathbb{C})$.

Theorem 3.1.15 (Bismut's Lichnerowicz formula)**.** *Let $\Delta^{B,\Lambda^{0,\cdot}\otimes E}$ be the Laplacian of $\nabla^{B,\Lambda^{0,\cdot}\otimes E}$ as in (3.1.1). Let r^M be the scalar curvature of M. We have*

$$\begin{aligned}2\square^E = 2(\bar{\partial}^E + \bar{\partial}^{E,*})^2 = \Delta^{B,\Lambda^{0,\cdot}\otimes E} + \frac{r^M}{4} + {}^c\left(R^E + \frac{1}{2}\operatorname{tr}\left[R^{T^{1,0}M}\right]\right) \\ + \frac{\sqrt{-1}}{2}\,{}^c(\partial\bar{\partial}\omega) - \frac{1}{8}|(\partial - \bar{\partial})\omega|^2.\end{aligned} \tag{3.1.63}$$

Remark that it generalises the Bochner-Kodaira for Kähler case.